

## Exercise 15

Solve the initial-value problem.

$$y'' + 4y' + 29y = 0, \quad y(0) = 1, \quad y(\pi) = -1$$

### Solution

This is a linear homogeneous ODE with constant coefficients, so it has solutions of the form  $y = e^{rx}$ .

$$y = e^{rx} \quad \rightarrow \quad y' = re^{rx} \quad \rightarrow \quad y'' = r^2e^{rx}$$

Substitute these formulas into the ODE.

$$r^2e^{rx} + 4(re^{rx}) + 29(e^{rx}) = 0$$

Divide both sides by  $e^{rx}$ .

$$r^2 + 4r + 29 = 0$$

Solve for  $r$ .

$$r = \frac{-4 \pm \sqrt{16 - 4(1)(29)}}{2} = \frac{-4 \pm \sqrt{-100}}{2} = -2 \pm 5i$$

$$r = \{-2 - 5i, -2 + 5i\}$$

Two solutions to the ODE are  $e^{(-2-5i)x}$  and  $e^{(-2+5i)x}$ . According to the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned} y(x) &= C_1e^{(-2-5i)x} + C_2e^{(-2+5i)x} \\ &= C_1e^{-2x}e^{-5ix} + C_2e^{-2x}e^{5ix} \\ &= e^{-2x}(C_1e^{-5ix} + C_2e^{5ix}) \\ &= e^{-2x}[C_1(\cos 5x - i \sin 5x) + C_2(\cos 5x + i \sin 5x)] \\ &= e^{-2x}[(C_1 + C_2)\cos 5x + (-iC_1 + iC_2)\sin 5x] \\ &= e^{-2x}(C_3 \cos 5x + C_4 \sin 5x) \end{aligned}$$

Apply the boundary conditions to determine  $C_3$  and  $C_4$ .

$$y(0) = C_3 = 1$$

$$y(\pi) = e^{-2\pi}(-C_3) = -1$$

Solving this system yields  $C_3 = 1$  and  $C_3 = e^{2\pi}$ , which is a contradiction. Therefore, there's no solution to the boundary value problem.